A mathematical view on the sockpuppet problem

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Abstract

We will consider two attempts to curtail the risk of sockpuppets by reducing the voting-weight of non-verified accounts to a small (but non-zero) value. It can be proven that at least in those cases where the non-verified accounts have an impact on the outcome of the vote, a malicious attacker can overrule *any* binary decision by simply adding a finite number of sockpuppets, even if the voting-weight of each non-verified account is reduced to a small but non-zero value or if the total voting-weight of all non-verified accounts is limited.

Disclaimer

Please note that allowing sockpuppets in any vote or in any decision-making system also violates the democratic principle of "one man – one vote". Therefore, the findings of this paper are only a supplementary reason for proper accreditation in democratic processes.

Conventions

- The symbol for the natural numbers $\mathbb{N}$ shall refer to a set that includes the number zero, i.e. $\mathbb{N} := \{0, 1, 2, 3, \ldots\}$.
- The symbol for rounding a number $r \in \mathbb{R}$ (or $r \in \mathbb{Q}$) down to the next integer value is $\lfloor r \rfloor$ such that $r \geq \lfloor r \rfloor \in \mathbb{Z}$ and $r - \lfloor r \rfloor < 1$.
- The symbol for rounding a number $r \in \mathbb{R}$ (or $r \in \mathbb{Q}$) up to the next integer value is $\lceil r \rceil$ such that $r \leq \lceil r \rceil \in \mathbb{Z}$ and $\lceil r \rceil - r < 1$.
- $\min(a, b) := a \text{ if } a \leq b \text{ else } b$
- $\max(a, b) := a \text{ if } a \geq b \text{ else } b$
- The term "sockpuppet" describes a fake identity created with malicious intent to manipulate public opinion and/or decisions by deception.
- Mr. Evil or Ms. Evil are used as names to refer to an attacker whose goal is to manipulate a decision by overruling the overall outcome of a vote using sockpuppets.

1 Reducing the voting-weight of non-verified accounts by a factor $x < 1$

We consider a voting system where internet users may create "non-verified" accounts on their own behalf (e.g. by using anonymous e-mail addresses or social-media accounts). These accounts may be legit or malicious (i.e. sockpuppets). A legit account is an account which is operated by a person who doesn't operate any other account in the system. In either case, each non-verified account gets a reduced voting-weight of $x < 1$, while users who verify their account through an accreditation process (with verification of identity) get a voting-weight of 1.

1.1 Conventions

P : The number of verified accounts (of which each has a voting-weight of 1)

x : Voting-weight of each non-verified account

n : The number of sockpuppets controlled by Mr. Evil

S : The number of all other (possibly legit) non-verified accounts

Consequently, the total voting-weight of all verified accounts is equal to P, whereas nx is the voting-weight of Mr. Evil, and Sx is the total voting-weight of all other non-verified accounts. Therefore, $P + Sx + nx$ is the total voting-weight of all accounts in the system.

1.2 Proposition 1

Let:

$$x > 0,\ P \in \mathbb{N},\ S \in \mathbb{N} \tag{1}$$

Then there exists an $n \in \mathbb{N}$ such that:

$$nx > \frac{P + Sx + nx}{2} \tag{2}$$

This means: For any (possibly very tiny) positive voting-weight $x > 0$ of each non-verified account and arbitrary counts of verified accounts P and (possibly legit) non-verified accounts S, there exists a number n of additional sockpuppets which Mr. Evil can add to gain a voting-weight of more than 50%, i.e. an absolute majority.

1.3 Proof of proposition 1

Choose n as follows:

$$\begin{aligned} n &:= \left\lfloor \frac{P}{x} \right\rfloor + S + 1 && (3) \\ &= \left\lfloor \frac{P}{x} + S \right\rfloor + 1 && (4) \end{aligned}$$

Note that $n \in \mathbb{N}$ because $\lfloor \frac{P}{x} + S \rfloor \in \mathbb{N}$. We further define $n' \in \mathbb{R}$:

$$n' := \frac{P}{x} + S \tag{5}$$

Then, because $x > 0$:

$$\begin{aligned}
& n = \lfloor n' \rfloor + 1 && (6)\\
\Rightarrow \quad & n > n' && (7)\\
\Leftrightarrow \quad & n > \frac{P}{x} + S && (8)\\
\Leftrightarrow \quad & n > \frac{P + Sx}{x} && (9)\\
\Leftrightarrow \quad & nx > P + Sx && (10)\\
\Leftrightarrow \quad & \frac{nx}{2} > \frac{P + Sx}{2} && (11)\\
\Leftrightarrow \quad & nx > \frac{P + Sx}{2} + \frac{nx}{2} && (12)\\
\Leftrightarrow \quad & nx > \frac{P + Sx + nx}{2} && (13)
\end{aligned}$$

Since $n \in \mathbb{N}$ and because (2) is identical to (13), proposition 1 is true. □

In other words: If Mr. Evil creates $n = \lfloor \frac{P}{x} \rfloor + S + 1$ sockpuppets, then he has more voting-weight than the other (verified and non-verified) accounts combined, i.e. he obtains an absolute majority by fraud, which empowers him to override the outcome of the vote.

2 Limiting the total voting-weight of all non-verified accounts to a constant value $T_{\max}$

In the following, we will consider another attempt to stop MR. (or MS.) EVIL from overruling majorities. We limit the *total* voting-weight of all non-verified accounts by reducing the voting-weight of each non-verified account proportionally as more non-verified accounts are created. Therefore, the total voting-weight of all non-verified accounts stays constant if $n \to \infty$.

As shown in this section, this attempt is also futile because whenever the non-verified accounts have an impact on the decision, a finite number of sockpuppets grants complete control over the outcome of the decision.

2.1 Conventions

P_{yes} : The number of verified accounts voting for "Yes"

P_{no} : The number of verified accounts voting for "No"

S_{yes} : The number of (possibly legit) non-verified accounts voting for "Yes", disregarding MS. EVIL's sockpuppets

S_{no} : The number of (possibly legit) non-verified accounts voting for "No", disregarding MS. EVIL's sockpuppets

n : The number of additional sockpuppets controlled by MS. EVIL

$T_{\max}$: Maximum total voting-weight of all non-verified accounts

T' : Total voting-weight of all non-verified accounts if MS. EVIL would not use her sockpuppets

T : Total voting-weight of all non-verified accounts if MS. EVIL manipulates the vote with her sockpuppets

Note that the voting options "yes" and "no" are chosen without loss of generality, i.e. "yes" / "no" could be replaced with "no" / "yes", "proposal A" / "proposal B", "status quo" / "amendment C", etc. Further note that the number of all non-verified accounts is $S_{\text{yes}}+S_{\text{no}}$ without MS. EVIL's sockpuppets and $S_{\text{yes}}+S_{\text{no}}+n$ with those sockpuppets.

2.2 Premises

Let:

$$T_{\max} > 0, \tag{14}$$

$$P_{\text{yes}} \in \mathbb{N},\ P_{\text{no}} \in \mathbb{N}, \tag{15}$$

$$S_{\text{yes}} \in \mathbb{N},\ S_{\text{no}} \in \mathbb{N} \tag{16}$$

We further premise that the non-verified accounts without $\text{M}\textsc{s}.\ \text{E}\textsc{vil}$'s sockpuppets have an actual impact on the overall outcome of the vote. Keeping in mind that "yes" and "no" were chosen without loss of generality, we do this by assuming:

$$P_{\text{yes}} > P_{\text{no}} \tag{17}$$

$$P_{\text{yes}} + T' \cdot \frac{S_{\text{yes}}}{S_{\text{yes}} + S_{\text{no}}} < P_{\text{no}} + T' \cdot \frac{S_{\text{no}}}{S_{\text{yes}} + S_{\text{no}}} \tag{18}$$

with

$$T' := \min(T_{\max}\,,\ S_{\text{yes}} + S_{\text{no}}) \tag{19}$$

being the total voting-weight of the non-verified accounts (excluding $\text{M}\textsc{s}.\ \text{E}\textsc{vil}$'s sockpuppets), and

$$S_{\text{yes}} + S_{\text{no}} > 0. \tag{20}$$

T' is defined in such way that it is $\leq S_{\text{yes}} + S_{\text{no}}$ (ensuring that each non-verified account gets a voting-weight of at most 1) but also limited by $T_{\max}$ (i.e. the total voting-weight of those accounts doesn't exceed $T_{\max}$). T' is then split up equally among all non-verified accounts (see inequality 18).

2.3 Proposition 2

Given the premises stated in the previous subsection 2.2, there exists an $n \in \mathbb{N}$ such that

$$P_{\text{yes}} + T \cdot \frac{S_{\text{yes}} + n}{S_{\text{yes}} + S_{\text{no}} + n} > P_{\text{no}} + T \cdot \frac{S_{\text{no}}}{S_{\text{yes}} + S_{\text{no}} + n} \tag{21}$$

with

$$T := \min(T_{\max}\,,\ S_{\text{yes}} + S_{\text{no}} + n) \tag{22}$$

being the effective total voting-weight of all non-verified accounts including $\text{M}\textsc{s}\ .\text{E}\textsc{vil}$'s sockpuppets.

This means: If we limit the total voting-weight T (or T' respectively) of all non-verified accounts to a constant but non-zero value $T_{\max}$ (inequality 14 with definitions 19 and 22), and split it up equally among all non-verified accounts, then, for any binary yes/no decision, there exists a number of additional sock-puppets n which Ms. Evil can add to overrule that decision (inequalities 18 and 21) if the other non-verified accounts $S_{\text{yes}} + S_{\text{no}} > 0$ had an impact on the overall outcome of the vote (inequalities 17 and 18).

2.4 Proof of proposition 2

From inequality (20) and $n \in \mathbb{N}$, we know that:

$$S_{\text{yes}} + S_{\text{no}} + n > 0 \tag{23}$$

We choose $n \in \mathbb{N}$ as follows:

$$n := S_{\text{no}} + 1 \tag{24}$$

Then inequality (21) can be transformed as follows:

$$\begin{aligned}
& P_{\text{yes}} + T \cdot \frac{S_{\text{yes}} + n}{S_{\text{yes}} + S_{\text{no}} + n} > P_{\text{no}} + T \cdot \frac{S_{\text{no}}}{S_{\text{yes}} + S_{\text{no}} + n} \\
\Leftrightarrow\ & P_{\text{yes}} + T \cdot \frac{S_{\text{yes}} + S_{\text{no}} + 1}{S_{\text{yes}} + S_{\text{no}} + n} > P_{\text{no}} + T \cdot \frac{S_{\text{no}}}{S_{\text{yes}} + S_{\text{no}} + n} \qquad (25) \\
\Leftrightarrow\ & P_{\text{yes}} + T \cdot \frac{S_{\text{yes}} + 1}{S_{\text{yes}} + S_{\text{no}} + n} > P_{\text{no}} \qquad (26)
\end{aligned}$$

Definition (22) with inequalities (14) and (23) implies that $T > 0$. Because (17) demands that $P_{\text{yes}} > P_{\text{no}}$, and (16) implies that $S_{\text{yes}} \geq 0$, we can easily see that inequality (26) is true. Therefore (21) is true. □

As shown in the following subsections, it is possible to provide another definition for n, which yields to an even smaller number of sockpuppets in many cases.

2.5 Proposition 3

The following alternative definition of n also fulfills inequality (21) with the given definition of T in (22) and the given premises in (14) through (20):

$$n := \max(\lfloor n' \rfloor + 1\,,\ \lceil T_{\max} \rceil) \tag{27}$$

with

$$n' := S_{\text{no}} \cdot \frac{1 - \dfrac{P_{\text{yes}} - P_{\text{no}}}{T_{\max}}}{1 + \dfrac{P_{\text{yes}} - P_{\text{no}}}{T_{\max}}} - S_{\text{yes}} \tag{28}$$

Note that n' is well-defined, because inequality (17) requires that $P_{\text{yes}} > P_{\text{no}}$ and (14) states that $T_{\max} > 0$.

2.6 Proof of proposition 3

From inequality (20) and $n \in \mathbb{N}$, we know that:

$$S_{\text{yes}} + S_{\text{no}} + n > 0 \tag{29}$$

Definition (22) with inequalities (14) and (29) implies that $T > 0$. Knowing $T > 0$, we use inequality (17) for the following estimation:

$$
\begin{aligned}
& P_{\text{yes}} > P_{\text{no}} \\
\Leftrightarrow \quad & P_{\text{yes}} - P_{\text{no}} > 0 && (30) \\
\Leftrightarrow \quad & \frac{P_{\text{yes}} - P_{\text{no}}}{T} > 0 && (31) \\
\Rightarrow \; 1 + & \frac{P_{\text{yes}} - P_{\text{no}}}{T} > 0 && (32)
\end{aligned}
$$

Definition (27) implies $n \geq T_{\max}$. Furthermore, it is presumed in (20) that $S_{\text{yes}} + S_{\text{no}} > 0$. Therefore:

$$
\begin{aligned}
& T_{\max} \leq n && (33) \\
\Rightarrow \quad & T_{\max} \leq S_{\text{yes}} + S_{\text{no}} + n && (34)
\end{aligned}
$$

From (22) and (34), it follows that:

$$T = T_{\max} \tag{35}$$

Definition (27) implies that $n > n'$. Using the definition of n' in (28), we reason:

$$n > \underbrace{S_{\text{no}} \cdot \frac{1 - \dfrac{P_{\text{yes}} - P_{\text{no}}}{T_{\max}}}{1 + \dfrac{P_{\text{yes}} - P_{\text{no}}}{T_{\max}}} - S_{\text{yes}}}_{n'} \tag{36}$$

$$\overset{(35)}{\Leftrightarrow} \quad n > S_{\text{no}} \cdot \frac{1 - \dfrac{P_{\text{yes}} - P_{\text{no}}}{T}}{1 + \dfrac{P_{\text{yes}} - P_{\text{no}}}{T}} - S_{\text{yes}} \tag{37}$$

$$\overset{(32)}{\Leftrightarrow} \quad n \cdot \left(1 + \frac{P_{\text{yes}} - P_{\text{no}}}{T}\right) > S_{\text{no}}\left(1 - \frac{P_{\text{yes}} - P_{\text{no}}}{T}\right) - S_{\text{yes}}\left(1 + \frac{P_{\text{yes}} - P_{\text{no}}}{T}\right) \tag{38}$$

$$\Leftrightarrow \quad n \cdot \left(1 + \frac{P_{\text{yes}} - P_{\text{no}}}{T}\right) > S_{\text{no}} - S_{\text{yes}} - \frac{P_{\text{yes}} - P_{\text{no}}}{T} \cdot (S_{\text{yes}} + S_{\text{no}}) \tag{39}$$

$$\Leftrightarrow \quad n + \frac{P_{\text{yes}} - P_{\text{no}}}{T} \cdot n > S_{\text{no}} - S_{\text{yes}} - \frac{P_{\text{yes}} - P_{\text{no}}}{T} \cdot (S_{\text{yes}} + S_{\text{no}}) \tag{40}$$

$$\Leftrightarrow \quad \frac{P_{\text{yes}} - P_{\text{no}}}{T} \cdot n > S_{\text{no}} - S_{\text{yes}} - \frac{P_{\text{yes}} - P_{\text{no}}}{T} \cdot (S_{\text{yes}} + S_{\text{no}}) - n \tag{41}$$

$$\Leftrightarrow \quad \frac{P_{\text{yes}} - P_{\text{no}}}{T} \cdot (S_{\text{yes}} + S_{\text{no}}) + \frac{P_{\text{yes}} - P_{\text{no}}}{T} \cdot n > S_{\text{no}} - S_{\text{yes}} - n \tag{42}$$

$$\Leftrightarrow \quad \frac{P_{\text{yes}} - P_{\text{no}}}{T} \cdot (S_{\text{yes}} + S_{\text{no}} + n) > S_{\text{no}} - (S_{\text{yes}} + n) \tag{43}$$

$$\overset{(29)}{\Leftrightarrow} \quad \frac{P_{\text{yes}} - P_{\text{no}}}{T} > \frac{S_{\text{no}} - (S_{\text{yes}} + n)}{S_{\text{yes}} + S_{\text{no}} + n} \tag{44}$$

$$\overset{T>0}{\Leftrightarrow} \quad P_{\text{yes}} - P_{\text{no}} > T \cdot \frac{S_{\text{no}} - (S_{\text{yes}} + n)}{S_{\text{yes}} + S_{\text{no}} + n} \tag{45}$$

$$\Leftrightarrow \quad P_{\text{yes}} - P_{\text{no}} > T \cdot \frac{S_{\text{no}}}{S_{\text{yes}} + S_{\text{no}} + n} - T \cdot \frac{S_{\text{yes}} + n}{S_{\text{yes}} + S_{\text{no}} + n} \tag{46}$$

$$\Leftrightarrow \quad P_{\text{yes}} + T \cdot \frac{S_{\text{yes}} + n}{S_{\text{yes}} + S_{\text{no}} + n} > P_{\text{no}} + T \cdot \frac{S_{\text{no}}}{S_{\text{yes}} + S_{\text{no}} + n} \tag{47}$$

Because inequalities (21) and (47) are identical and (27) ensures $n \in \mathbb{N}$, we conclude that proposition 3 is true. □

In other words: If Ms. Evil creates $n = \max(\lfloor n' \rfloor + 1,\, \lceil T_{\max} \rceil)$ additional sockpuppets (definition 27) and if the other non-verified accounts would be relevant for the outcome of the vote if Ms. Evil didn't use her sockpuppets (inequalities 17 and 18), then Ms. Evil can overrule the overall outcome of the vote (inequalities 18 and 21).

Note that we could still construct cases where Ms. Evil can't gain control over the outcome of the vote. In those cases, however, the non-verified accounts would not have any effect on the outcome of the vote anyway, which is why non-verified accounts could have been excluded from casting votes in the first place.

In *all* cases where non-verified accounts have an actual impact on the outcome of the vote, manipulation is possible by adding a finite number of sockpuppets. It is therefore futile to try to curtail the sockpuppet problem by limiting the total voting-weight of all non-verified accounts.

2.7 Example

We choose the following example where inequalities 17 and 18 are fulfilled, i.e. where the non-verified voters $S_{\text{yes}} + S_{\text{no}}$ have an impact on the outcome:

$$\begin{aligned} P_{\text{yes}} &= 503 \quad & S_{\text{yes}} &= 48 \\ P_{\text{no}} &= 497 \quad & S_{\text{no}} &= 952 \\ T_{\max} &= 10 \end{aligned}$$

According to proposition 3, it is certain that Ms. Evil can control the outcome of the vote with 191 sockpuppets:

$$\begin{aligned} n &= \max(\lfloor n' \rfloor + 1\,,\ \lceil T_{\max} \rceil) \\ &= \max\left(\left\lfloor S_{\text{no}} \cdot \frac{1 - \frac{P_{\text{yes}} - P_{\text{no}}}{T_{\max}}}{1 + \frac{P_{\text{yes}} - P_{\text{no}}}{T_{\max}}} - S_{\text{yes}} \right\rfloor + 1\,,\ \lceil T_{\max} \rceil\right) \\ &= \max\left(\left\lfloor 952 \cdot \frac{1 - \frac{503 - 497}{10}}{1 + \frac{503 - 497}{10}} - 48 \right\rfloor + 1\,,\ \lceil 10 \rceil\right) \\ &= \max\left(\left\lfloor 952 \cdot \frac{1 - \frac{6}{10}}{1 + \frac{6}{10}} - 48 \right\rfloor + 1\,,\ 10\right) \\ &= \max\left(\left\lfloor 952 \cdot \frac{4/10}{16/10} - 48 \right\rfloor + 1\,,\ 10\right) \\ &= \max\left(\left\lfloor 952 \cdot \frac{1}{4} - 48 \right\rfloor + 1\,,\ 10\right) \\ &= \max\left(\lfloor 238 - 48 \rfloor + 1\,,\ 10\right) \\ &= \max\left(\lfloor 190 \rfloor + 1\,,\ 10\right) \\ &= \max\left(191\,,\ 10\right) \\ &= 191 \end{aligned}$$

3 Conclusion

All voting systems where internet users may create non-verified accounts on their own behalf are also susceptible to the creation of “sockpuppets”. Sockpuppets are fake identities created with malicious intent to manipulate public opinion and/or decisions by deception. Often, the reduction of barriers is brought up as an argument in favor of easy account creation and against proper accreditation systems (i.e. proper user identification and verification).

However, it can be proven that at least in those cases where the non-verified accounts have an impact on the outcome of the vote, a malicious attacker can overrule *any* binary decision by simply adding a finite number of sockpuppets, even if the voting-weight of each non-verified account is reduced to a small but non-zero value (section 1 of this paper) or if the total voting-weight of all non-verified accounts is limited (section 2 of this paper).

Since creation of sockpuppets already violates the democratic principle of “one man – one vote”, these findings are only a supplementary reason for proper accreditation in democratic processes. Refer to the book “The Principles of LiquidFeedback” (ISBN 978–3–00–044795–2), section *“Who may participate? (And how are these people identified?)”* (pages 120–124) for further reading.

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